CS 598: Spectral Graph Theory. Lecture 2

Extremal Eigenvalues and Eigenvectors of the Laplacian and the Adjacency Matrix.

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Today

- Bounding eigenvalues
- Courant-Fischer and Rayleigh quotients
- Applications of Courant-Fischer
- Adjacency matrix vs. Laplacian
- Chromatic number
- Perron-Frobenius

Start Bounding Laplacian Eigenvalues

Sum of Eigenvalues, Extremal Eigenvalues

- $\sum_i \lambda_i = \sum_i d_i \leq d_{max} n$ where di is the degree of vertex i.

  Proof: take the trace of L

- $\lambda_2 \leq \frac{\sum_i d_i}{n-1}$ and $\lambda_n \geq \frac{\sum_i d_i}{n-1}$

  Proof: previous inequality + $\lambda_1 = 0$.

Courant-Fischer

- For any nxn symmetric matrix A with eigenvalues $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ (decreasing order) and corresponding eigenvectors $v_1, v_2, \dots, v_n$, denote S_k the span of $v_1, v_2, \dots, v_k$ and ${S_k}^{\perp}$ the orthogonal complement , then

$$\alpha_k = \max_{x \in S_{k-1}^{\perp}, x \neq 0} \frac{x^T A x}{x^T x} \qquad \alpha_1 = \max_{x \neq 0} \frac{x^T A x}{x^T x}$$

Proof: see blackboard

Courant-Fischer

- **Courant-Fischer Min Max Formula:** For any nxn symmetric matrix A with eigenvalues $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ (decreasing order),

$$\alpha_k = \max_{S \subseteq R^n, \dim(S)=k} \min_{x \in S} \frac{x^T A x}{x^T x}$$

$$\alpha_k = \min_{S \subseteq R^n, \dim(S)=n-k+1} \max_{x \in S} \frac{x^T A x}{x^T x}$$

Proof: see blackboard

Courant-Fischer for Laplacian

- Courant-Fischer Min Max Formula for increasing evalue order (e.g. Laplacians): For any nxn symmetric matrix L, with eigenvalues in increasing order

$$\lambda_k = \min_{S \text{ of } \dim k} \max_{x \in S} \frac{x^T L x}{x^T x}$$

$$\lambda_k = \max_{S \text{ of } \dim n-k-1} \min_{x \in S} \frac{x^T L x}{x^T x}$$

- Definition (Rayleigh Quotient): The ratio $\frac{x^T L x}{x^T x}$ is called the *Rayleigh Quotient* of x with respect to L.
- We will use it to bound evalues of Laplacians of certain graphs.

Courant-Fischer for Laplacian

$$\lambda_k = \max_{S \subseteq R^n, \dim(S) = n-k+1} \min_{x \in S} \frac{x^T L x}{x^T x}$$

- Applying Courant-Fischer for the Laplacian

we get :

$$\lambda_1 = 0, v_1 = 1$$

$$\lambda_2 = \min_{x \perp 1, x \neq 0} \frac{x^T L x}{x^T x} = \min_{x \perp 1, x \neq 0} \frac{\sum_{(i,j) \in E} (x_i - x_j)^2}{\sum_{i \in V} x_i^2}$$

$$\lambda_{\max} = \max_{x \neq 0} \frac{x^T L x}{x^T x} = \max_{x \neq 0} \frac{\sum_{(i,j) \in E} (x_i - x_j)^2}{\sum_{i \in V} x_i^2}$$

- Useful for getting bounds, if calculating spectra is cumbersome.
- To get upper bound on λ_2, just need to produce vector with small Rayleigh Quotient.
- Similarly, t o get lower bound on λ_{max}, just need to produce vector with large Rayleigh Quotient

Example 1

- *Lemma1:* Let G=(V,E) be a graph with some vertex w having degree d. Then

$$\lambda_{\max} \geq d$$

- *Lemma 2:* We can also improve on that. Under same assumptions, we can show:

$$\lambda_{\max} \geq d + 1$$

Proof: see blackboard

Example 1

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Lemma 2 is tight, take star graph (ex)

Example 2

- The Path graph P_n on n vertices has

$$\lambda_2 \le \frac{12}{n^2}$$

- Already knew that eigenvalues are

$2 - 2\cos\left(\frac{\pi k}{n}\right) \approx 2\left(1 - 1 + \frac{\pi^2}{2n^2}\right) \approx \frac{\pi^2}{n^2}$, but this is easier and more general.

Proof: see blackboard

Example 3

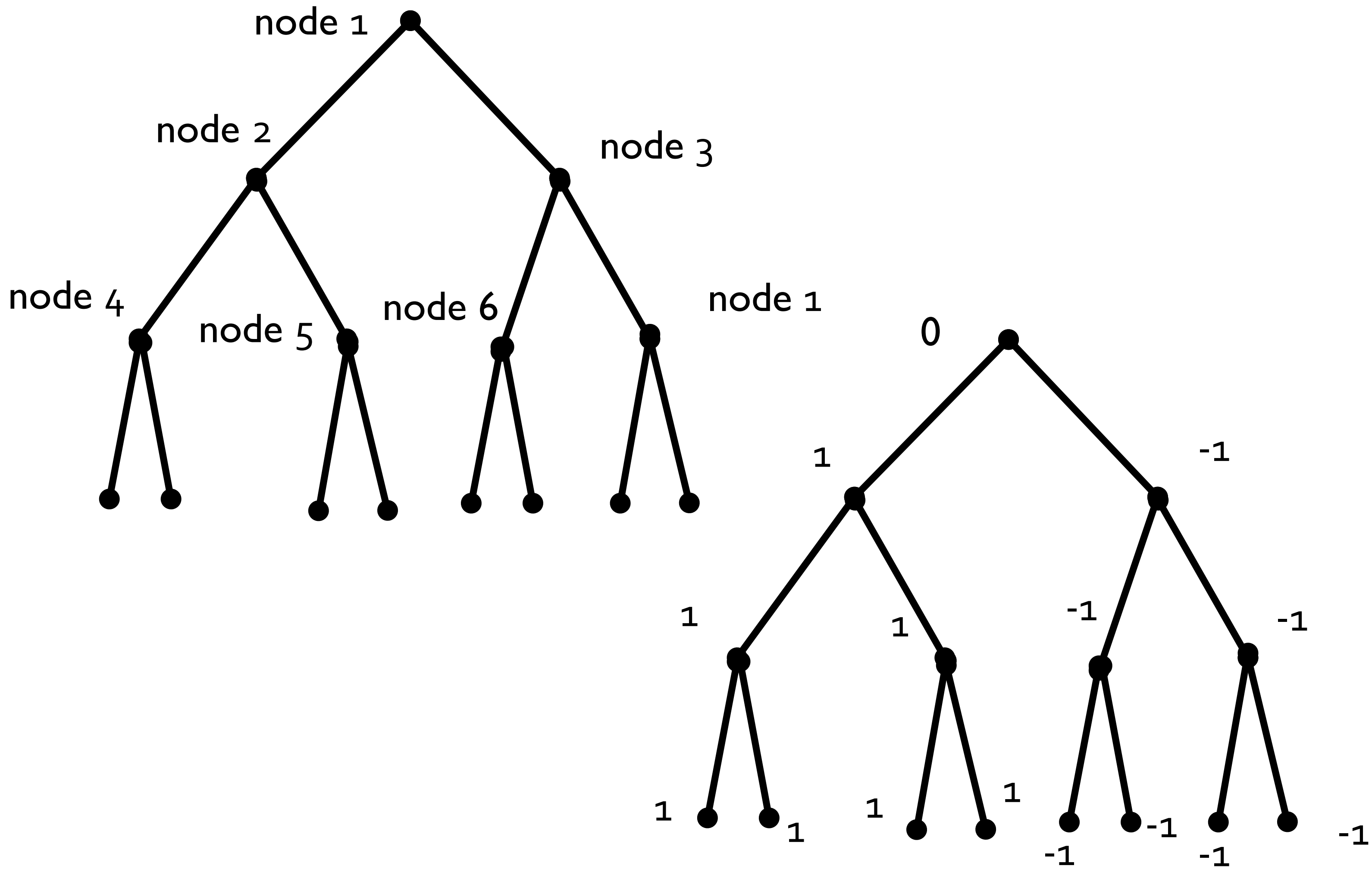
- The complete binary tree B_n on n =$2^d - 1$ vertices has

$$\lambda_2 \leq \frac{2}{n-1}$$

B_n is the graph with edges of the form (u,2u) and (u,2u+1) for u<n/2.

Proof: See blackboard

Example 3


node 1
node 2
node 3
node 4
node 5
node 6
node 1
0
1
-1
1
1
-1
-1
1
1
1
1
1
-1
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-1
-1
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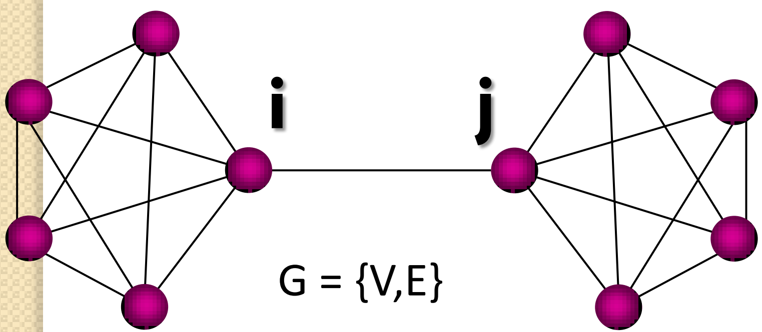
- Lower bounds are harder, we will see some in two lectures (different technique)

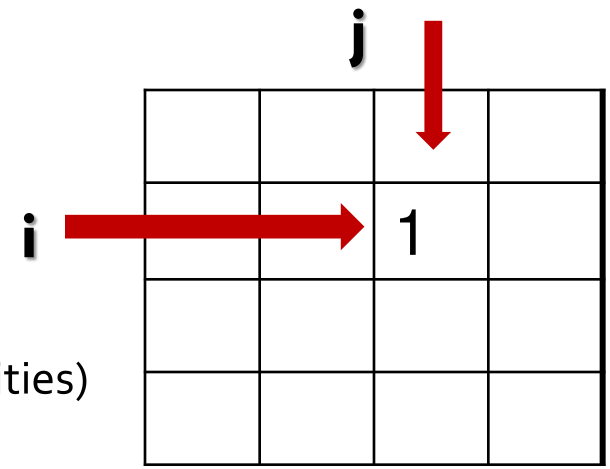
Adjacency Matrix vs. Laplacian

Adjacency Matrix Refresher



$$A_{ij} = \begin{cases} 1, & if\ (i,j)\ edge \\ 0 & if\ no\ edge\ between\ \ i,j \end{cases}$$

- Unweighted graphs for simplicity



A has n eigenvalues (counting multiplicities)
$\{a_1 \geq a_2 \geq \ldots \geq a_n\}$

- Adjacency matrix as operator:
  $(A_G \boldsymbol{u})(i) = \sum_{j:(i,j)\in E} \boldsymbol{v}(j)$

Adjacency Matrix vs. Laplacian for d-regular graphs

- G is d-regular if every vertex has degree d. In this case: $L_G = D_G - A_G = dI - A_G$
- Let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the evalues of L and $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ be the evalues of A.
- We have $\alpha_i = d - \lambda_i$ and the corresponding evectors are the same.

Bounds on the Eigenvalues of Adjacency Matrix

- $\alpha_1 \le d_{max}$

Proof: See blackboard.

- Adjacency matrix as operator:
  $(A_G \boldsymbol{u})(i) = \sum_{j:(i,j)\in E} \boldsymbol{v}(j)$

Bounds on the Eigenvalues of Adjacency Matrix

- $\alpha_1 \leq d_{max}$ with equality iff graph is d_{max} – regular. In this case, the first eigenvector is the all-one's vector. (exercise)

Courant-Fischer for Adjacency Matrix Refresher

$$\alpha_k = \max_{S \text{ of } \dim k} \min_{x \in S} \frac{x^T A x}{x^T x}$$

$$\alpha_1 = \max_{x \neq 0} \frac{x^T A x}{x^T x}$$

- Will see next how to apply Courant-Fischer for the adjacency matrix to get another bound on the first eigenvalue as well as a relation to graph coloring

Bounding Adjacency Matrix Eigenvalues

- **Lemma 1:** α_1 is at least the average degree of the vertices in G

  Proof: see blackboard

$$\alpha_1 = \max_{x \neq 0} \frac{x^T A x}{x^T x}$$

Bounding Adjacency Matrix Eigenvalues

- **Lemma 1:** α_1 is at least the average degree of the vertices in G

- While we may think of α_1 as being related to the average degree, it behaves differently. If we remove the vertex of smallest degree in a graph, the average degree can increase. However, α_1 only decreases when we remove a vertex.

- **Lemma2:** Let A be a symmetric matrix, let B be the matrix obtained by removing the last row and column from A and let b_1 be the largest eigenvalue of B. Then $\alpha_1 \geq b_1$

Proof: see blackboard

Chromatic Number

- The chromatic number of a graph G, denoted $\chi(G)$, is the least k for which G has a k-coloring.

- **Theorem (Wilf):** $\chi(G) \leq \lfloor \alpha_1 \rfloor + 1$

Proof: see blackboard

Chromatic Number

- The chromatic number of a graph G, denoted $\chi(G)$, is the least k for which G has a k-coloring.

- **Theorem (Wilf):** $\chi(G) \leq \lfloor \alpha_1 \rfloor + 1$

- Improvement over classical bound $\chi(G) \leq d_{max}+1$, as there are graphs (e.g. path graph) where α_1 is much less than d_{max}

Adjacency Matrix: The Perron-Frobenius Theorem

- We saw what happens for regular graphs. What is G is not regular? We know that $\alpha_1 < d_{max}$ but what about v_1?

Perron-Frobenius Theorem (for graphs): Let G=(V,E,w) be a connected graph, A its adjacency matrix and $\mu_1 \geq \cdots \geq \mu_n$ its evalues. Then:

(i) $\mu_1 \geq -\mu_n$

(ii) $\mu_1 > \mu_2$

(iii) μ_1 has a strictly positive eigenvector

Bipartite?

- Another graph Property and eigenvalues: Bipartiteness
- ***Theorem***: If G is a connected graph then $\mu_1 = -\mu_n$ iff G is bipartite.

Coloring, again, and Independent Sets

- The most negative eigenvalue of the adjacency matrix (and the largest eigenvalue of the Laplacian) corresponds to the highest frequency vibration in a graph. Its eigenvector tries to assign as different as possible values to neighbors. Corresponds to coloring.

Coloring, again, and Independent Sets

- Theorem (Hoffman). Let S be an independent set in G, and let dav(S) be the average degree of a vertex in S. Then

$$|S| \leq n(1 - \frac{d_{av}(S)}{\lambda_n})$$

- It follows that $\chi_G \geq \frac{\lambda_n}{\lambda_n - d_{av}}$ (exercise)

Laplacian: The Perron-Frobenius Theorem

- Theory can also be applied to Laplacians and any matrix with non-positive off-diagonal entries. It involves the eigenvector with smallest eigenvalue.

Perron-Frobenius for Laplacians:Let M be a matrix with non-positive off-diagonal entries s.t. the graph of the no-zero off-diagonal entries is connected. Then the smallest eigenvalue has multiplicity 1 and the corresponding eigenvector is strictly positive.

Next time we will see how to apply Peron Frobenius to show Fiedler's nodal domain theorem.